

EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH ZETA FUNCTIONS

TAEKYUN KIM

ABSTRACT. For $s \in \mathbb{C}$, Euler zeta function and Hurwitz's type Euler zeta function are defined by

$$\zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \text{ and } \zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \text{ see [20, 21, 22, 31, 15].}$$

Thus, we note that Euler zeta functions are entire functions in whole complex s -plane and these zeta functions have the values of Euler numbers or Euler polynomials at negative integers. That is,

$$\zeta_E(-k) = E_k^*, \text{ and } \zeta_E(-k, x) = E_k^*(x), \text{ cf. [12, 20, 21, 22, 31].}$$

In this paper, we give some interesting identities between Euler numbers and zeta functions. Finally we will give the new values of Euler zeta function at positive even integers.

§1. Introduction

Throughout this paper \mathbb{Z} , \mathbb{Q} , \mathbb{C} , \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will respectively denote the ring of rational integers, the field of rational numbers, the field of complex numbers, the ring p -adic rational integers, the field of p -adic rational numbers, and the completion of the algebraic closure of \mathbb{Q}_p . Let v_p be the normalized exponential valuation of \mathbb{C}_p such that $|p|_p = p^{-v_p(p)} = p^{-1}$. If $q \in \mathbb{C}_p$, we normally assume $|q - 1|_p < 1$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Hence, $\lim_{q \rightarrow 1} [x]_q = 1$, for any x with $|x|_p \leq 1$ in the present p -adic case.

Key words and phrases. Bernoulli numbers, twisted Bernoulli numbers, zeta function, Euler numbers, twisted Euler numbers, q -Volkenborn integral, Dedekind sum, p -adic invariant integral.

2000 AMS Subject Classification: 11B68, 11S80

Let p be a fixed odd prime. For $d(= \text{odd})$ a fixed positive integer with $(p, d) = 1$, let

$$\begin{aligned} X = X_d &= \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^n}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

In this paper we prove that

$$\mu_{-q}(a + dp^N\mathbb{Z}_p) = (1 + q) \frac{(-1)^a q^a}{1 + q^{dp^N}} = \frac{(-q)^a}{[dp^N]_{-q}},$$

is distribution on X for $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. This distribution yields an integral as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad \text{for } f \in UD(\mathbb{Z}_p),$$

which has a sense as we see readily that the limit is convergent (see [14]). Let $q = 1$. Then we have the fermionic p -adic integral on \mathbb{Z}_p as follows:

$$I_{-1} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{cf. [1, 5, 20, 21, 14].}$$

For any positive integer N , we set

$$\mu_q(a + lp^N\mathbb{Z}_p) = \frac{q^a}{[lp^N]_q}, \quad \text{cf. [1, 5, 6, 7, 10, 11-18, 27-31],}$$

and this can be extended to a distribution on X . This distribution yields p -adic bosonic q -integral as follows (see [13, 31]):

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_X f(x) d\mu_q(x),$$

where $f \in UD(\mathbb{Z}_p) =$ the space of uniformly differentiable function on \mathbb{Z}_p with values in \mathbb{C}_p , cf.[1, 13, 27, 28, 29, 30, 31]. In view of notation, I_{-1} can be written symbolically as $I_{-1}(f) = \lim_{q \rightarrow -1} I_q(f)$. If we take $f(x) = q^{-x}[x]_q^n$, then we can derive the q -extension of Bernoulli numbers and polynomials from p -adic q -integrals on \mathbb{Z}_p as follows:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} q^{-x}[x]_q^n d\mu_q(x), \quad \text{and } \beta_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y}[y+x]_q d\mu_q(y), \quad \text{cf. [13, 31].}$$

Thus, we note that

$$\beta_{0,q} = \frac{q-1}{\log q}, \text{ and } \beta_{m,q} = \frac{1}{(q-1)^m} \sum_{i=0}^m \binom{m}{i} \frac{i}{[i]_q}, \text{ cf. [13, 17, 31].}$$

In complex plane, the ordinary Bernoulli numbers are a sequence of signed rational numbers that can be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi, \text{ cf. [1-33].}$$

These numbers arise in the series expansions of trigonometric functions, and are extremely important in number theory and analysis. From the generating function of Bernoulli numbers, we note that $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, B_{16} = -\frac{3617}{510}, B_{18} = \frac{43867}{798}, B_{20} = -\frac{174611}{330}, \dots$, and $B_{2k+1} = 0$ for $k \in \mathbb{N}$. It is well known that Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ for } s \in \mathbb{C}.$$

We also note that Riemann zeta function is closely related to Bernoulli numbers at positive integer or negative integer in complex plane. Riemann did develop the theory of analytic continuation needed to rigorously define $\zeta(s)$ for all $s \in \mathbb{C} - \{0\}$. From this zeta function, he derived the following formula, cf. [1-33]:

$$\zeta(-n) = -\frac{B_n + 1}{n + 1}, \quad n \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

Thus, we note that $\zeta(-n) = 0$ if n is even integer and greater than 0. These are called the trivial zeros of the zeta function. In 1859, starting with Euler's factorization of the zeta function

$$\zeta(s) = \prod_{p: \text{prime}} \frac{1}{1 - p^{-s}},$$

he derived an explicit formula for the prime numbers in terms of zeros of the zeta function. He also posed the Riemann Hypothesis: if $\zeta(z) = 0$, then either z is a trivial zero or z lies on the critical line $\text{Re}(z) = \frac{1}{2}$, cf. [20-33]. It is well known that

$$\frac{\sin z}{z} = \left(1 - \frac{z^2}{\pi^2}\right) \left(1 - \frac{z^2}{(2\pi)^2}\right) \left(1 - \frac{z^3}{(3\pi)^2}\right) \dots.$$

Thus,

$$1 - z \cot z = 2 \sum_{m=1}^{\infty} \frac{\zeta(2m)}{\pi^{2m}} z^{2m}, \text{ cf. [12, 20-33].}$$

From this, we can derive the following famous formula:

Lemma. For $n \in \mathbb{N}$, we have

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} (2\pi)^{2n}}{2(2n)!} B_{2n}, \text{ for } n \in \mathbb{N}.$$

Since

$$z \cot z = \frac{2iz}{e^{2iz} - 1} + iz = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} z^{2k}.$$

However, it is not known the values of $\zeta(2k+1)$ for $k \in \mathbb{N}$. In the case of $k = 1$, Apery proved that $\zeta(3)$ is irrational number. The constants E_k^* in the Taylor series expansion

$$(1) \quad \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \text{ where } |t| < \pi, \text{ cf. [5, 12, 20, 21, 22]},$$

are known as the first kind Euler numbers. From the generating function of the first kind Euler numbers, we note that

$$E_0^* = 1, \text{ and } E_n^* = - \sum_{l=0}^n \binom{n}{l} E_l^*, \text{ for } n \in \mathbb{N}.$$

The first few are $1, -\frac{1}{2}, 0, \frac{1}{4}, \dots$, and $E_{2k}^* = 0$ for $k = 1, 2, \dots$. The Euler polynomials are also defined by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} E_k^* x^{n-k} \right) \frac{t^n}{n!}.$$

For $s \in \mathbb{C}$, Euler zeta function and Hurwitz's type Euler zeta function are defined by

$$\zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \text{ and } \zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \text{ cf. [1, 11, 12, 20, 21, 22].}$$

Thus, we note that Euler zeta functions are entire functions in whole complex s -plane and these zeta functions have the values of Euler numbers or Euler polynomials at negative integers. That is,

$$\zeta_E(-k) = E_k^*, \text{ and } \zeta_E(-k, x) = E_k^*(x), \text{ cf. [1, 11, 12, 20, 21, 22].}$$

In this paper, we give some interesting identities between Euler numbers and zeta functions. Finally we will give the values of Euler zeta function at positive even integers.

§2. Preliminaries/Euler numbers associated with p -adic fermionic integrals

Let $f_1(x)$ be translation with $f_1(x) = f(x + 1)$. Then we have

$$I_{-1}(f_1) = I_{-1}(f) + 2f(0).$$

If we take $f(x) = e^{(x+y)t}$, then we can derive the first kind Euler polynomials from the integral equation of $I_{-1}(f)$ as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = e^{xt} \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n^*(x)t^n}{n!}.$$

That is,

$$\int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = E_n^*, \text{ and } \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n^*(x).$$

For $n \in \mathbb{N}$, we have the following integral equation:

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^l f(l).$$

From this we note that

$$\begin{aligned} E_k^*(n) - E_k^* &= 2 \sum_{l=0}^{n-1} (-1)^{l-1} l^k, \text{ if } n \equiv 0 \pmod{2}, \\ E_k^*(n) + E_k^* &= 2 \sum_{l=0}^{n-1} (-1)^l l^k, \text{ if } n \equiv 1 \pmod{2}. \end{aligned}$$

Let $f(x) = \sin ax$ (or $f(x) = \cos ax$). By using the fermionic p -adic q -integral on \mathbb{Z}_p , we see that

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) \\ &= (\cos a + 1) \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) + \sin a \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x), \text{ see [15].} \end{aligned}$$

and

$$2 = (\cos a + 1) \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) - \sin a \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x).$$

Thus, we obtain

$$\int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) = 1, \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = -\frac{\sin a}{\cos a + 1}, \text{ see [15].}$$

From this we note that

$$\tan \frac{a}{2} = - \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a^{2n+1}}{(2n+1)!} E_{2n+1}^*.$$

By the same motivation, we can also observe that

$$\frac{a}{2} \cot \frac{a}{2} = \int_{\mathbb{Z}_p} \cos ax d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} a^{2n}, \text{ see [15].}$$

These formulae are also treated in the Section 3.

Let $f(x) = e^{t(2x+1)}$. Then we can derive the generating function of the second kind Euler numbers from fermionic p -adic integral equation as follows:

$$\int_{\mathbb{Z}_p} e^{t(2x+1)} d\mu_{-1}(x) = \frac{2}{e^t + e^{-t}} = \frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Thus, we have

$$(E+1)^n + (E-1)^n = 2\delta_{0,n},$$

where we have used the symbolic notation E_n for E^n . The first few are $E_0 = 1, E_1 = 0, E_2 = -1, E_3 = 0, E_4 = 5, \dots, E_{2k+1} = 0$ for $k \in \mathbb{N}$. In particular,

$$E_{2n} = - \sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}.$$

In the recent, Simsek, Ozden, Cangul, Cenkci, Kurt, etc have studied the various extensions of the first kind Euler numbers by using fermionic p -adic invariant q -integral on \mathbb{Z}_p , see [1, 5, 20, 21, 22, 27, 31]. It seems to be also interesting to study the q -extensions of the second kind Euler numbers due to Simsek et al(see [20, 21, 27]).

§3. Some relationships between Euler numbers and zeta functions

In this section we also consider Bernoulli and the second Euler numbers in complex plane. The second kind Euler numbers E_k are defined by the following expansion:

$$(2) \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2e^x}{e^{2x} + 1} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}, \text{ for } |t| < \frac{\pi}{2}, \text{ cf.[12].}$$

From (1) and (2), we can derive the following equation:

$$(3) \quad E_k = \sum_{l=0}^k \binom{k}{l} 2^l E_l^*, \text{ where } \binom{k}{l} \text{ is binomial coefficient.}$$

By (3) and (1), we easily see that $E_0 = 1, E_1 = 0, E_2 = -1, E_3 = 0, E_4 = 5, E_6 = 61, \dots$, and $E_{2k+1} = 0$ for $k = 1, 2, 3, \dots$. As Euler formula, it is well known that

$$(4) \quad e^{ix} = \cos x + i \sin x, \text{ where } i = (-1)^{\frac{1}{2}}.$$

From (4), we note that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$. Thus, we have

$$(5) \quad \begin{aligned} \sec x &= \frac{2}{e^{ix} + e^{-ix}} = \operatorname{sech}(ix) = \sum_{n=0}^{\infty} \frac{i^n E_n}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n+1}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}. \end{aligned}$$

From (5), we derive

$$(6) \quad x \sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n+1}, \text{ for } |x| < \frac{\pi}{2}.$$

The Fourier series of an odd function on the interval $(-p, p)$ is the sine series:

$$(7) \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right),$$

where

$$(8) \quad b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

Let us consider $f(x) = \sin ax$ on $[-\pi, \pi]$. From (7) and (8), we note that

$$(9) \quad \sin ax = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$(10) \quad \begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin ax \sin nx dx = \frac{2}{\pi} \int_0^{\pi} \left[\frac{\cos(n-a)x - \cos(n+a)x}{2} \right] dx \\ &= \frac{1}{\pi} \left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a} \right]_0^{\pi} = (-1)^{n-1} \frac{2}{\pi} \sin a\pi \left(\frac{n}{n^2 - a^2} \right). \end{aligned}$$

In (9), if we take $x = \frac{\pi}{2}$, then we have

$$(11) \quad \begin{aligned} \sin \frac{\pi a}{2} &= \sum_{n=1}^{\infty} b_{2n-1} (-1)^{n-1} \\ &= \frac{2}{\pi} \sin a\pi \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n-1}{(2n-1)^2 - a^2} = \frac{2}{\pi} \sin a\pi \sum_{n=1}^{\infty} \frac{(2n-1)(-1)^{n-1}}{(2n-1)^2 \left(1 - \left(\frac{a}{2n-1}\right)^2\right)} \\ &= \frac{2}{\pi} \sin a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sum_{k=0}^{\infty} \frac{a^{2k}}{(2n-1)^{2k}} = \frac{2}{\pi} \sin a\pi \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} \right) a^{2k} \end{aligned}$$

From (11), we note that

$$(12) \quad \frac{\pi a}{2} \sec\left(\frac{\pi a}{2}\right) = \sum_{k=0}^{\infty} \left(2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} \right) a^{2k+1}.$$

In (6), it is easy to see that

$$(13) \quad \frac{\pi a}{2} \sec\left(\frac{\pi a}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} \left(\frac{\pi}{2}\right)^{2n+1} a^{2n+1}.$$

By (12) and (13), we obtain the following:

Theorem 1. *For $n \in \mathbb{N}$, we have*

$$(14) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2n+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}.$$

It is easy to see that

$$(15) \quad \begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} &= 2 \sum_{n=1}^{\infty} \frac{1}{(4n-3)^{2k+1}} + \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k+1}} - \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} - 1 \\ &= \frac{1}{2^{4k+1}} \zeta(2k+1, \frac{1}{4}) - \frac{2^{2k+1}-1}{2^{2k+1}} \zeta(2k+1) - 1. \end{aligned}$$

By (14) and (15), we obtain the following:

Corollary 2. *For $n \in \mathbb{N}$, we have*

$$\zeta(2n+1, \frac{1}{4}) + 2^{2n} (1 - 2^{2n+1}) \zeta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!} \pi^{2n+1} 2^{2n}.$$

By simple calculation, we easily see that

$$i \tan x = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = 1 - \frac{2}{e^{2ix} - 1} + \frac{4}{e^{4ix} - 1}.$$

Thus, we have

$$(16) \quad \begin{aligned} x \tan x &= -xi + \frac{2xi}{e^{2xi} - 1} - \frac{4xi}{e^{4xi} - 1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n} 4^n (1 - 4^n)}{(2n)!} x^{2n}. \end{aligned}$$

From (16), we can easily derive

$$(17) \quad \tan x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^{n+1} (1 - 4^{n+1}) B_{2n+2}}{(2n+2)!} x^{2n+1}.$$

By (4), we also see that

$$i \tan x = 1 - \frac{2}{e^{2ix} + 1} = i \sum_{n=0}^{\infty} \frac{E_{2n+1}^*}{(2n+1)!} 2^{2n+1} (-1)^{n+1}.$$

Thus, we have

$$(18) \quad \tan x = \sum_{n=0}^{\infty} \frac{E_{2n+1}^*}{(2n+1)!} 2^{2n+1} (-1)^{n+1} x^{2n+1}.$$

By (17) and (18), we obtain the following:

Theorem 3. *For $n \in \mathbb{N}$, we have*

$$\zeta(2n) = \frac{(-1)^{n-1} (2\pi)^{2n} E_{2n-1}^*}{4(2n-1)! (1 - 4^n)},$$

where E_n^* are the first kind Euler numbers.

It is easy to see that

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2n}} = (1 - \frac{1}{4^n}) \zeta(2n) = \frac{(-1)^n (2\pi)^{2n}}{4^{n+1} (2n-1)!} E_{2n-1}^*.$$

Therefore we obtain the following:

Corollary 4. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2n}} = \frac{(-1)^n (2\pi)^{2n}}{4^{n+1} (2n-1)!} E_{2n-1}^*.$$

Now we try to give the new value of Euler zeta function at positive integers. From the definition of Euler zeta function, we note that

$$(19) \quad \zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} + \frac{1}{2^{s-1}} \zeta(s), \quad s \in \mathbb{C}.$$

By (19), Theorem 3 and Corollary 4, we obtain the following:

Theorem 5. For $n \in \mathbb{N}$, we have

$$\zeta_E(2n) = \frac{(-1)^{n-1} \pi^{2n} (2 - 4^n)}{2(2n-1)!(1-4^n)} E_{2n-1}^*.$$

Remark. We note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta_E(2) = -\frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta_E(4) = -\frac{7\pi^4}{360} \dots$. For $q \in \mathbb{C}$ with $|q| < 1$, $s \in \mathbb{C}$, q - ζ -function is defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s} - \frac{1}{s-1} \frac{(1-q)^s}{\log q}, \text{ cf. [12, 17].}$$

Note that, $\zeta_q(s)$ is analytic continuation in \mathbb{C} with only one simple pole at $s = 1$, and

$$\zeta_q(1-k) = -\frac{\beta_{k,q}}{k}, \text{ where } k \text{ is a positive integer, cf. [17].}$$

By simple calculation, we easily see that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^{2k+1}} \sum_{j=0}^{\infty} \frac{\theta^{2j+1} [n]_q^{2j+1}}{(2j+1)!} + \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{(1-q)^{2k-2j}}{(2k-2j-1)(2j+1)!} \\ & - \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{(1-q)^{2k-2j}}{(2k-2j-1)(2j+1)!} \\ & = \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left(-\zeta_q(2k-2j) + \zeta_{q^2}(2k-2j) \frac{2}{[2]_q^{2k-2j}} \right) \\ & - \frac{q}{1+q} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k + \sum_{j=k+1}^{\infty} \frac{\theta^{2j+1} (-1)^j}{(2j+1)!} \frac{H_{2j-2k,q}(-q^{-1})}{1+q}, \end{aligned}$$

where $H_{n,q}(-q)$ are Carlitz's q -Euler numbers with $\lim_{q \rightarrow 1} H_{n,q}(-q) = E_n^*$, cf. [2, 3, 6]. If $q \rightarrow 1$, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k+1}} \sin(n\theta) &= \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \theta^{2j+1} \left(\frac{2}{2^{2k-2j}} - 1 \right) \\ & (-1)^{k-j+1} \frac{(2\pi)^{2k-2j}}{2 \cdot (2k-2j)!} B_{2k-2j} - \frac{1}{2} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k. \end{aligned}$$

For $k \in \mathbb{N}$, and $\theta = \frac{\pi}{2}$, it is easy to see that

$$(20) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^{2k+1}} = \sum_{j=0}^{k-1} \frac{(-1)^j \pi^{2k+1} (2^{2k-2j} - 2) B_{2k-2j}}{(2j+1)!(2k-2j)! 2^{2j+2}} - \frac{\pi^{2k+1} (-1)^k}{(2k+1)! 2^{2k+2}}.$$

From (20) and Theorem 1, we can also derive the following equation (21).

$$(21) \quad \sum_{j=0}^{k-1} \frac{(-1)^{k-1} \pi^{2k+1} (2^{2k-2j} - 2) B_{2k-2j}}{(2j+1)!(2k-2j)!2^{2j+2}} + \frac{\pi^{2k+1} (-1)^k}{(2k+1)!2^{2k+2}} = (-1)^k \frac{E_{2k}}{2(2k)!} \left(\frac{\pi}{2}\right)^{2k+1}.$$

Thus, we have

$$\sum_{j=0}^{k-1} \frac{(2^{2k-2j} - 2) B_{2k-2j}}{(2j+1)!(2k-2j)!2^{2j+2}} = \frac{1}{(2k+1)!2^{2k+2}} - \frac{E_{2k}}{2^{2k+2}(2k)!}.$$

REFERENCES

- [1] I. N. Cangul, V. Kurt, Y. Simsek, H.K. Pak, S.-H. Rim, *An invariant p-adic q-integral associated with q-Euler numbers and polynomials*, J. Nonlinear Math. Phys. **14** (2007), 8–14.
- [2] L. C. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.
- [3] L. C. Carlitz, *Expansions of q-Bernoulli numbers*, Duke Math. J. **25** (1958), 355–364.
- [4] L. C. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc. **76** (1954), 332–350.
- [5] M. Cenkci, *The p-adic generalized twisted (h, q)-Euler-l-function and its applications*, Advan. Stud. Contemp. Math. **15** (2007), 37–47.
- [6] M. Cenkci, M. Can and V. Kurt, *p-adic interpolation functions and Kummer-type congruences for q-twisted Euler numbers*, Advan. Stud. Contemp. Math. **9** (2004), 203–216.
- [7] M. Cenkci, M. Can, *Some results on q-analogue of the Lerch zeta function*, Adv. Stud. Contemp. Math. **12** (2006), 213–223.
- [8] E. Deeba, D. Rodriguez, *Stirling's series and Bernoulli numbers*, Amer. Math. Monthly **98** (1991), 423–426.
- [9] A.S. Hegazi, M. Mansour, *A note on q-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys. **13** (2006), 9–18.
- [10] Y. Jang, H. Kim, *A series whose terms are products of two q-Bernoulli numbers in the p-adic case*, Houston J. Math. **27** (2001), 495–510.
- [11] T. Kim, *On p-adic q-l-functions and sums of powers*, J. Math. Anal. Appl. **329** (2007), 1472–1481.
- [12] T. Kim, L.C. Jang, S.H. Rim, S.C. Park, J.Y. Sug, J. Y. Choi, C. S. Ryoo, *Introduction to Non-Archimedean Integrals and Their Applications*, Kyo Woo Sa, 2007.
- [13] T. Kim, *q–Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288–299.
- [14] T. Kim, *A Note on p-Adic q-integral on \mathbb{Z}_p Associated with q-Euler Numbers*, Adv. Stud. Contemp. Math. **15** (2007), 133–138.
- [15] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on \mathbb{Z}_p at $q = -1$* , J. Math. Anal. Appl. **331** (2007), 779–792.
- [16] T. Kim, *q-Extension of the Euler formula and trigonometric functions*, Russ. J. Math. Phys. **14** (2007), 275–278.
- [17] T. Kim, *Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function*, Russ. J. Math. Phys. **12** (2005), 186–196.
- [18] T. Kim, *Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russ. J. Math. Phys. **10** (2003), 91–98.
- [19] B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), 412–422.
- [20] H. Ozden, Y. Simsek, I.N. Cangul, *Euler polynomials associated with p-adic q-Euler measure*, General Mathematics **15** (2007), 24–37.

- [21] H. Ozden, Y. Simsek, *A new extension of q -Euler numbers and polynomials related to their interpolation functions*, Appl. Math. Lett. (2007), doi:10.1016/j.aml.2007.10.005.
- [22] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, *A note on p -adic q -Euler measure*, Adv. Stud. Contemp. Math. **14** (2007), 233–239.
- [23] C.S. Ryoo, *The zeros of the generalized twisted Bernoulli polynomials*, Adv. Theor. Appl. Math. **1** (2006), 143–148.
- [24] M. Schork, *Ward's "calculus of sequences", q -calculus and the limit $q \rightarrow -1$* , Adv. Stud. Contemp. Math. **13** (2006), 131–141.
- [25] M. Schork, *Combinatorial aspects of normal ordering and its connection to q -calculus*, Adv. Stud. Contemp. Math. **15** (2007), 49–57.
- [26] K. Shiratani, S. Yamamoto, *On a p -adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A **39** (1985), 113–125.
- [27] Y. Simsek, *On twisted q -Hurwitz zeta function and q -two-variable L -function*, Appl. Math. Comput. **187** (2007), 466–473.
- [28] Y. Simsek, *On p -adic twisted q - L -functions related to generalized twisted Bernoulli numbers*, Russ. J. Math. Phys. **13** (2006), 340–348.
- [29] Y. Simsek, *Twisted (h, q) -Bernoulli numbers and polynomials related to twisted (h, q) -zeta function and L -function*, J. Math. Anal. Appl. **324** (2006), 790–804.
- [30] Y. Simsek, *Theorems on twisted L -function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. **11** (2005), 205–218.
- [31] Y. Simsek, *q -Dedekind type sums related to q -zeta function and basic L -series*, J. Math. Anal. Appl. **318** (2006), 333–351.
- [32] H.J.H. Tuenter, *A Symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly **108** (2001), 258–261.
- [33] John. C. Baez, *The Riemann zeta function*, Preprint (December 21, 2003) (2003).

Taekyun Kim
 EECs, Kyungpook National University,
 Taegu 702-701, S. Korea
 e-mail: tkim@knu.ac.kr; tkim64@hanmail.net